J. Fluid Mech. (1965), *vol.* 23, *part* 3, *pp.* 471–479
Printed in Great Britain

Effects of viscosity on ship waves

By E. CUMBERBATCH
Mathematics Department, University of Leeds†

(Received 11 January 1965 and in revised form 12 April 1965)

The effect of viscosity on the trailing-wave system well downstream of a uniformly moving surface disturbance is determined. Solutions are obtained by series expansions in inverse powers of a Reynolds number and coefficients in the viscous exponential decay-factor found. It is seen that the diverging wave pattern is much more heavily damped than the transverse waves.

1. Introduction

The waves set up by a body moving at uniform velocity over the free surface of a fluid have previously been described by inviscid theory (e.g. see Wehausen & Laitone 1960). The equations of motion and the boundary conditions are linearized and it is possible, therefore, to obtain the solution for a pressure distribution representing the body from the solution for a moving pressure point. The purpose of this paper is to incorporate viscosity into the solution of the latter problem, in particular as it affects the waves far from the pressure point. It is not intended to make any comments on the effect of applying a real fluid boundary condition at the body.

Some methods of estimating the wave resistance of ships and other surface craft are based on measurements of the trailing wave profiles created by towing models. The scaling involved requires these estimates to be as accurate as possible. The results of the present work make it possible to compare the wave profiles including and excluding viscosity, and so to make the adjustments necessary in the resistance calculations which are based on the inviscid model.

The effect of viscosity on ship waves has been investigated in detail for the two-dimensional case by Wu & Messick (1958). The method involves the use of Fourier transforms and solutions are found in the form of expansions close to and far from the body at low and high Reynolds numbers. The extension to the three-dimensional case requires double Fourier integrals. The expansions of the solutions in the various limiting cases become complicated because of the algebraic nature of the functions and also because of the estimation of the accuracy of the solution obtained. For these reasons only the case of wave profiles at large distances downstream for high Reynolds numbers is investigated. This is the case of particular interest for the applications mentioned above.

The three-dimensional wave systems set up by a moving pressure point in the inviscid case were investigated by Lord Kelvin (Sir W. Thomson 1891). Recent improvements of the solution are due to Peters (1949) and Ursell (1960).

† Now at the Division of Mathematical Sciences, Purdue University, Lafayette, Indiana.

Crapper (1964) has shown how the results for the wave system well downstream can be easily obtained using the methods for the asymptotic expansion of Fourier integrals described by Lighthill (1960). The present paper continues with this latter scheme for linearized equations which include the viscous terms. Formal expansions in inverse powers of a Reynolds number are obtained, together with estimates of error terms. These expansions have a different form inside and on the critical lines (where the radial decay factor changes). However, a viscous decay factor is obtained which is valid over the transverse and diverging wave systems up to and including the critical lines.

2. Solution as a Fourier integral

Cartesian co-ordinates are taken fixed in the body with the fluid at large distances having velocity U in the x-direction. The z-direction is vertically upwards. The linearized (Oséen) equations of motion are

$$\nabla . \mathbf{q} = 0, \tag{1}$$

$$U\frac{\partial \mathbf{q}}{\partial x}+\frac{1}{\rho}\nabla(p+\rho g z)-\nu\nabla^2\mathbf{q} = 0, \tag{2}$$

where
$$U\mathbf{i}+\mathbf{q} = (U+u, v, w) \tag{3}$$

is the total velocity, ρ is the density and ν the kinematic viscosity. These equations allow a splitting of the form
$$\mathbf{q} = \mathbf{q}_1+\mathbf{q}_2, \tag{4}$$

where
$$\operatorname{curl}\mathbf{q}_1 = 0, \quad \operatorname{div}\mathbf{q}_2 = 0, \tag{5}$$

such that
$$U\frac{\partial \mathbf{q}_1}{\partial x}+\frac{1}{\rho}\nabla(p+\rho g z) = 0, \tag{6}$$

$$U\frac{\partial \mathbf{q}_2}{\partial x}-\nu\nabla^2\mathbf{q}_2 = 0. \tag{7}$$

It follows from (5) that
$$\mathbf{q}_1 = \nabla\phi \quad \text{with} \quad \nabla^2\phi = 0. \tag{8}$$

Also, from (6)
$$p/\rho = -U\frac{\partial \phi}{\partial x}-gz. \tag{9}$$

The boundary conditions are

$$p/\rho-2\nu\frac{\partial w}{\partial z} = -U\frac{\partial \phi}{\partial x}-g\eta-2\nu\left(\frac{\partial^2\phi}{\partial z^2}+\frac{\partial w_2}{\partial z}\right) = -\frac{1}{\rho}G, \tag{10}$$

$$\nu\left(\frac{\partial u_2}{\partial z}+2\frac{\partial^2\phi}{\partial x\,\partial z}+\frac{\partial w_2}{\partial x}\right) = \frac{1}{\rho}H, \tag{11}$$

$$\nu\left(\frac{\partial v_2}{\partial z}+2\frac{\partial^2\phi}{\partial y\,\partial z}+\frac{\partial w_2}{\partial y}\right) = \frac{1}{\rho}K, \tag{12}$$

and
$$U\frac{\partial \eta}{\partial x} = \frac{\partial \phi}{\partial z}+w_2. \tag{13}$$

These boundary conditions are to be applied on $z = 0$, where

$$z = \eta(x,y) \tag{14}$$

is the equation of the free surface. Surface tension has been neglected. In (10), (11) and (12) the functions G, H, K represent the stress distribution acting on the free surface, G being the applied normal stress, and H, K the x and y components of the shearing stress. In what follows both H and K will be taken to be zero, but results for non-vanishing shearing stresses can be obtained without difficulty.

Denoting the Fourier transform with respect to x and y of $f(x,y,z)$ by $\tilde{f}(\alpha,\beta,z)$, equations (7) and (8) yield

$$\tilde{\phi} = A(\alpha,\beta)\exp[(\alpha^2+\beta^2)^{\frac{1}{2}}z], \tag{15}$$

$$\tilde{u}_2 = B(\alpha,\beta)\exp[(\alpha^2+\beta^2+i\alpha U/\nu)^{\frac{1}{2}}z], \tag{16}$$

with similar expressions to (16) for $\tilde{v}$, $\tilde{\omega}_2$. The square roots in (15), (16) are required to have positive real parts for α, β real, to ensure the vanishing of the perturbation solution as $z \to -\infty$. Substituting (15), (16) and the expressions for v_2, w_2 into the boundary conditions (10)–(13), together with (5), allows the constants A, B, etc., to be found in terms of G, H, K. The result for $\tilde{\eta}$ in the case $H = K = 0$ is

$$\tilde{\eta} = \frac{\tilde{G}}{\rho}\{g-(\alpha^2+\beta^2)^{\frac{1}{2}}[U\alpha(\alpha^2+\beta^2)^{-\frac{1}{2}}-2\nu i(\alpha^2+\beta^2)^{\frac{1}{2}}]^2 \\ -4\nu^2(\alpha^2+\beta^2)(\alpha^2+\beta^2+i\alpha U/\nu)^{\frac{1}{2}}\}^{-1} \tag{17}$$

and the height of the free surface is obtained as the inverse Fourier transform of (17).

The case of a normal stress of delta-function form at the origin is taken here. This gives $\tilde{G} = G_0$, const. The solution for a distribution of pressure can be obtained by further integration. It is convenient to introduce non-dimensional variables

$$\bar{x} = x/l, \quad \bar{y} = y/l, \quad \bar{\alpha} = \alpha l, \quad \bar{\beta} = \beta l, \tag{18}$$

where
$$l = U^2/g. \tag{19}$$

A Reynolds number
$$R = Ul/\nu = U^3/\nu g \tag{20}$$

is defined. The wave profile $z = \eta(x,y)$ is now given by the double Fourier integral

$$\eta = (4\pi^2l^2\rho g)^{-1}G_0 I = (4\pi^2l^2\rho g)^{-1}G_0\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\alpha\bar{x}+\beta\bar{y})}D^{-1}\,d\alpha\,d\beta, \tag{21}$$

where

$$D = 1-(\alpha^2+\beta^2)^{\frac{1}{2}}[\alpha(\alpha^2+\beta^2)^{-\frac{1}{2}}-2i(\alpha^2+\beta^2)^{\frac{1}{2}}/R]^2-4R^{-2}(\alpha^2+\beta^2)(\alpha^2+\beta^2+iR\alpha)^{\frac{1}{2}}. \tag{22}$$

The expansion of (21) for large $(x^2+y^2)^{\frac{1}{2}}$ and for large R is evaluated in the next section. Actually the expansions are made for large x and it requires additional analysis to make the results valid for large $(x^2+y^2)^{\frac{1}{2}}$. Since only regions where $y < \frac{1}{4}x\sqrt{2}$ are of interest, these details will be neglected.

3. Expansion of the solution

Methods for the asymptotic expansion of Fourier integrals similar to (21) have been described by Lighthill. Ship wave profiles neglecting viscosity but including surface tension have been obtained by Crapper using these methods.

In essence, the method involves calculating the α-integration by residues and the β-integration by stationary phase. The poles of the integrand give a contribution representing the steady wave-form well downstream. The effect of the inclusion of viscosity is to move the poles off the real axis; viscous damping is thereby obtained. Also the stationary-phase method used for stationary points on the real axis has to be replaced by a steepest-descents analysis. It is seen that the equation yielding the poles cannot be solved exactly and only approximate locations of the poles can be obtained. The analysis is carried through to yield an approximate solution in terms of an expansion at large distances and at large Reynolds numbers. It becomes important to estimate the accuracy of the approximation and to determine its region of validity.

Contributions to the α-integration in (21) arise from the singularities of D. The function $D(\alpha)$ has branch points at

$$\alpha = \pm i|\beta| \quad \text{and at} \quad \alpha = \tfrac{1}{2}i[-R \pm (R^2+4\beta^2)^{\frac{1}{2}}], \tag{23}$$

and the branch cuts are taken from $\alpha = i|\beta|$ to $\alpha = \frac{1}{2}i[(R^2+4\beta^2)^{\frac{1}{2}} - R]$ on the positive $\operatorname{Im}(\alpha)$ axis, and from $\alpha = -i|\beta|$ to $\alpha = -\frac{1}{2}i[(R^2+4\beta^2)^{\frac{1}{2}}+R]$ on the negative $\operatorname{Im}(\alpha)$ axis. The analytic nature of D is investigated further by putting

$$\alpha = s\cos\psi, \quad \beta = s\sin\psi, \tag{24}$$

giving

$$D = 1 - s(\cos\psi - 2is/R)^2 - 4s^2R^{-2}(s^2+isR\cos\psi)^{\frac{1}{2}}. \tag{25}$$

It is noted that D has the same analytic form as its two-dimensional counterpart which was dealt with by Wu. The techniques used there can be repeated to show that D has two simple zeros. For large R these zeros are found at

$$\alpha = \alpha_1^* = A + iB/R + O(R^{-\frac{3}{2}}) \tag{26}$$

and

$$\alpha = \alpha_2^* = -A + iB/R + O(R^{-\frac{3}{2}}),$$

where

$$A = \frac{1}{\sqrt{2}}\{1+(1+4\beta^2)^{\frac{1}{2}}\}^{\frac{1}{2}}, \tag{27}$$

$$B = 4A^6(2A^2-1)^{-1}. \tag{28}$$

The inviscid case gives zeros at $\alpha = \pm A$. The location of the zeros α_1^*, α_2^* in the upper half-plane insures that there is no wave contribution for $x < 0$. (The real-axis integration is replaced by a semi-circular contour of large radius in the lower half-plane.) The inviscid case requires other arguments to achieve this result.

For $x > 0$, the path of the α-integration in (21) is replaced by a large semi-circular contour in the upper half α-plane and it is necessary to consider the contributions from the singularities of D. The branch point singularities contribute integrals taken along the branch cuts in the upper half α-plane indicated by (23). For the purposes of estimating the order of magnitude of these integrals. their limiting values as $R \to \infty$ may be taken in the first approximation. Integrals of the form

$$\int_{-\infty}^{\infty} e^{i\beta y}\left\{\int_0^{i|\beta|} e^{i\alpha x}[1-\alpha(\alpha^2+\beta^2)^{-\frac{1}{2}}]^{-1}\,d\alpha\right\}d\beta \tag{29}$$

are then apparent and it is not difficult to show that they are of order x^{-3} (see Ursell, 1960).

The zeros of D contribute the dominant terms in the expansion at large distances. For the moment, we take the first two terms on the right-hand side of (26), that is $\alpha_1 = A + iB/R$ to give the location of the zero. The approximations inherent in this procedure are discussed later. Also, only the contribution from the zero at $\alpha = \alpha_1$ is considered up to the point at which a physical result is written down. This zero gives a contribution

$$I_1 = 2\pi i \int_{-\infty}^{\infty} \exp\{rf(\beta)\} [(\partial D/\partial\alpha)_{\alpha_1}]^{-1} \, d\beta, \tag{30}$$

in which $f(\beta) = i\{\beta \sin\theta + \cos\theta (A(\beta) + iB(\beta)/R)\}$ and

$$x = r\cos\theta, \quad y = r\sin\theta, \tag{31}$$

are to be substituted.

A steepest-descents approximation for large r is now calculated. The cols in the β-plane are determined from the equation

$$\sin\theta + (A' + iB'/R)\cos\theta = 0, \tag{32}$$

where accents denote differentiation with respect to β. In the inviscid limit, (32) reduces to

$$A' = -\tan\theta. \tag{33}$$

This may be interpreted geometrically by stating that (33) is satisfied by points on the curve

$$\alpha = A(\beta), \tag{34}$$

at which the normal to the curve is parallel to the vector $\mathbf{r} = (r\cos\theta, r\sin\theta)$. As a result, geometrical information about the wave pattern can be obtained from the nature of the curves given by (34) (e.g. see Lighthill and Crapper). Solutions of (33) are possible only for $|\theta| \leqslant \theta_c$, where $\tan\theta_c = \frac{1}{4}\sqrt{2}$. The critical points, denoted by the subscript c, are the inflexion points, $\beta_c^2 = \frac{3}{4}$, where $A''(\beta_c) = 0$. Equation (33) has two solutions for each θ, giving rise to two wave systems: the transverse system for $\beta^2 < \frac{3}{4}$ and the diverging system for $\beta^2 > \frac{3}{4}$. Different radial decay laws are in force for the wave systems in regions close to and away from the lines $\theta = \pm\theta_c$. This can be appreciated by considering the steepest-descents calculation which will involve $A''(\beta)$. A method which gives uniformly valid results over these regions is described by Ursell.

The effects of viscosity may be determined by solving (32) in terms of a series in inverse powers of R. The cases $A'' = 0$ and $A'' \neq 0$ become separate calculations in a straightforward expansion procedure. However, these calculations may be combined by using Ursell's method and the extension of this method for a steepest-descents calculation for (30) is contained in what follows. The results for $\theta = \pm\theta_c$ and θ not near $\pm\theta_c$ are readily deduced and will be quoted. Ursell's notation will be used.

A change of variable given by

$$A(\beta) = (1 + u^2)^{\frac{1}{2}} \tag{35}$$

is introduced by which $f(\beta)$ can be written

$$f(\beta(u)) = ig(u) = i\{g_0(u,\theta) + g_1(u,\theta)/R\},$$

where
$$\left.\begin{aligned} g_0 &= \{\cos\theta - u\sin\theta\}(1+u^2)^{\frac{1}{2}}, \\ g_1 &= 4i(1+u^2)^3(2u^2+1)^{-1}\cos\theta. \end{aligned}\right\} \tag{36}$$

The function $g(u)$ is now replaced by a cubic polynomial. That is, the substitution
$$g(u) = -\tfrac{1}{3}v^3 + \mu(\theta, R)\,v - \nu(\theta, R) \tag{37}$$
is made to enable (30) to be computed in terms of the Airy function. For (37) to represent a regular transformation, it is required that $du/dv \neq 0$. However, dg/du is to vanish at the col locations for the steepest-descents calculation and it becomes necessary to match the zeros of dg/du with $v = \pm\mu^{\frac{1}{2}}$.

There are two solutions to (33) for $|\theta| < \theta_c$. Denote solutions to (33) by β_0^+ and β_0^-. Let the corresponding zeros of dg_0/du be denoted by u_0^+, u_0^- and the zeros of dg/du by u_1^+, u_1^-. That is, dg/du is $O(R^{-\frac{3}{2}})$ at $u = u_1^\pm$. The matching of the zeros gives
$$\begin{aligned} \nu &= -\tfrac{1}{2}\{g_0(u_0^+) + g_0(u_0^-)\} - \tfrac{1}{2}\{g_1(u_0^+) + g_1(u_0^-)\}/R \\ &= \nu_0 - \tfrac{1}{2}\{g_1(u_0^+) + g_1(u_0^-)\}/R \end{aligned} \tag{38}$$
and
$$\begin{aligned} \tfrac{2}{3}\mu^{\frac{3}{2}} &= \tfrac{1}{2}\{g_0(u_0^+) - g_0(u_0^-)\} + \tfrac{1}{2}\{g_1(u_0^+) - g_1(u_0^-)\}/R \\ &= \tfrac{2}{3}\mu_0^{\frac{3}{2}} + \tfrac{1}{2}\{g_1(u_0^+) - g_1(u_0^-)\}/R, \end{aligned}$$
where u_0, ν_0 are functions evaluated by Ursell. Equations (38) are obtained by expanding $g_0(u_1^+)$, etc., and using the result that $(dg_0/du) = 0$ at $u = u_0^\pm$. Hence, μ and ν can be obtained as series representations and (38) display the first two terms.

The transformation (37) allows I_1 to be expressed in terms of Airy functions as
$$I_1 \sim 4\pi^2 \exp(-ir\nu)\{ir^{-\frac{1}{3}}p_0 A_i(-r^{\frac{2}{3}}\mu) - r^{-\frac{2}{3}}q_0 A_i'(-r^{\frac{2}{3}}\mu)\}, \tag{39}$$
where $p_0(\theta)$, $q_0(\theta)$ are evaluated by Ursell. The expressions (38) for μ, ν remain to be substituted in (39) and the asymptotic form found from the properties of the Airy function. The main effect of viscosity is now evident in (39). The viscous-dependent terms in the expansions (38) for μ and ν have imaginary argument and (39) gives rise to an exponential viscous decay factor. For example, in regions away from the critical lines, the asymptotic form is obtained by expanding the Airy functions for large values of $|r^{\frac{2}{3}}\mu|$ and, retaining the viscous terms only in the exponential factors, this yields
$$\begin{aligned} I_1 \sim 2\pi^{\frac{3}{2}}r^{-\frac{1}{2}}\mu_0^{-\frac{1}{4}}[(p_0 + \mu_0^{\frac{1}{2}}q_0)\exp i\{\tfrac{2}{3}\mu_0^{\frac{3}{2}}r - \nu_0 r + g_1(u_0^+)\,r/R - \tfrac{1}{4}\pi\} + (p_0 - \mu_0^{\frac{1}{2}}q_0) \\ \times \exp i\{-\tfrac{2}{3}\mu_0^{\frac{3}{2}}r - \nu_0 r + g_1(u_0^-)\,r/R + \tfrac{1}{4}\pi\}]. \end{aligned} \tag{40}$$
The first and second terms in braces in (40) give the transverse and diverging wave contributions which are damped by factors
$$\exp(ig_1(u_0^+)\,r/R), \quad \exp(ig_1(u_0^-)\,r/R),$$
respectively. The viscous decay factor is thus $\exp(-B_0\cos\theta r/R)$ and the nature of its dependence on θ is investigated in § 4.

It is desirable to estimate the errors introduced by the series development (38) in the asymptotic evaluation (39). In this respect, the asymptotic calculations applied directly to (30) enable the errors to be estimated more easily than from (39). This procedure will be carried out for the cases θ not near $\pm\theta_c$ and $\theta = \pm\theta_c$.

Case 1: θ not near $\pm\theta_c$

The first approximation to the solution of (32) is given by

$$\beta = \beta_1 = \beta_0 - iB_0'\{A_0''R\}^{-1}, \tag{41}$$

where $B_0' = B'(\beta_0)$, etc. Since the regions near the critical points are being avoided, A_0'' is not close to a zero and (41) is a valid expansion. At the stationary points $\beta = \beta_1$, the expansion up to the term in R^{-1} for f'' is given by

$$f'' = i\cos\theta\{A_0'' + iR^{-1}(B_0'' - A_0'''B_0'/A_0'')\},$$

or

$$f'' = Ce^{i\gamma},$$

where

$$\gamma = \pi - \tfrac{1}{2}\pi\operatorname{sgn}A_0'' + R^{-1}(B_0'/A_0'')', \tag{42}$$

$$C = |A_0''\cos\theta|\{1 + O(R^{-2})\}.$$

The asymptotic approximation of I_1 for large r can now be written as

$$I_1 = 2\pi i\Sigma(2\pi/Cr)^{\frac{1}{2}}\exp\{rf(\beta_1) + i(\tfrac{1}{2}\pi - \tfrac{1}{2}\gamma)\}[(\partial D/\partial\alpha)_{\alpha_1(\beta_1)}]^{-1} + O(1/r), \tag{43}$$

where the summation is over the roots β_1. This expansion for I_1 may be obtained from (39) using asymptotic forms for the Airy functions for large $|r^{\frac{2}{3}}\mu|$.

The approximations inherent in obtaining the result given by (43) will now be discussed. Let I be the integral corresponding to (30) using α_1^* instead of α_1 to give the location of the zeros. Consider substitutions $t = \alpha_1^*(\beta)$ or $\alpha_1(\beta)$ to be made in the integrals I, I_1, respectively. Solutions for β of the form

$$\beta = \beta_0(t) + \beta_1(t)/R + \beta_2(t)/R^{\frac{3}{2}} + \dots \tag{44}$$

will result and will differ only from the third term onwards depending on the choice α_1^* or α_1. It can be seen then that

$$(I - I_1)/I = O(R^{-\frac{3}{2}}) \tag{45}$$

and the use of α_1 in place of α_1^* in (30) introduces error terms in (44) of $O(r^{-\frac{1}{2}}R^{-\frac{3}{2}})$.

The asymptotic representation (43) is a formal expansion for large R of the exact asymptotic value of (30). That is, it is assumed that the position of the cols can be found exactly. Let the exact col position corresponding to β_1 be denoted by β_E. The exact asymptotic expansion of (30) is obtained by substituting β_E for β_1 in (43). The dominant error term appears in the exponential which can be written

$$\exp\{rf(\beta_E)\} = \exp\{rf(\beta_1) + rf'(\beta_1)(\beta_E - \beta_1) + \dots\}. \tag{46}$$

Since $(\beta_E - \beta_1)$ and $f'(\beta_1)$ are both $O(R^{-\frac{3}{2}})$, the error terms in (44) are $O(rI_1R^{-3})$.

The asymptotic expansion for the wave profile (21) is found by adding contributions similar to (43) for both zeros α_1 and α_2. This gives

$$\eta \sim \frac{iG_0}{2\pi\rho g l^2}\Sigma\left(\frac{2\pi}{rC}\right)^{\frac{1}{2}}\exp(-rR^{-1}B_0\cos\theta)$$
$$\times\left[\exp\{ih(\beta)\}\left\{\left(\frac{\partial D}{\partial\alpha}\right)_{\alpha_1}\right\}^{-1} + \exp\{-ih(\beta)\}\left\{\left(\frac{\partial D}{\partial\alpha}\right)_{\alpha_2}\right\}^{-1}\right], \tag{47}$$

where $h(\beta) = \alpha\bar{x} + \beta\bar{y} + \frac{1}{4}\pi\operatorname{sgn}A_0'' - \frac{1}{2}R^{-1}(B_0'/A_0'')'$ and the summation is taken over points where the normal to $\alpha = A(\beta)$ for $\alpha > 0$ is parallel to r. The expression (47) has error terms of order r^{-1}, $r^{\frac{1}{2}}R^{-3}$ and $r^{-\frac{1}{2}}R^{-\frac{3}{2}}$.

Of interest is the exponential decay in (47) since this is the major effect of viscosity on the inviscid wave profiles. The leading term in (47) is

$$\eta \sim \frac{-G_0}{\pi\rho g l^2}\Sigma\left\{\frac{2\pi}{r|K|}\right\}^{\frac{1}{2}}\cos\theta\exp(-rR^{-1}B_0\cos\theta)\sin(\alpha\bar{x}+\beta\bar{y}+\tfrac{1}{4}\pi\operatorname{sgn}K)\left[\left(\frac{\partial D}{\partial\alpha}\right)_{A_0}\right]^{-1}, \tag{48}$$

where $K = [A_0''\cos^3\theta]$ is the curvature of $\alpha = A(\beta)$. The terms omitting the exponential term in (48) constitute the inviscid solution. The result (48) would be obtained from (21) under the assumption that the stationary points for the inviscid solution do not change in the first approximation.

Case 2: $\theta = \pm\theta_c$

At the critical points, where $A''(\beta_0) = 0$, the expansion (41) breaks down and the expansion in this case has the first two terms

$$\beta = \beta_1 = \beta_0 + \delta_1 R^{-\frac{1}{2}},$$

where
$$\delta_1 = \pm(-2iB_0'/A_0''')^{\frac{1}{2}}. \tag{49}$$

With this choice of δ_1, $f'(\beta_1)$ is $O(R^{-\frac{3}{2}})$ and $f''(\beta_1)$ is $O(R^{-\frac{1}{2}})$. The integral I_1 has now the form

$$I_1 = 2\pi i\int_{-\infty}^{\infty}\exp\{rf(\beta_1)+\tfrac{1}{6}rf'''(\beta_1)(\beta-\beta_1)^3+\ldots\}\left[\left(\frac{\partial D}{\partial\alpha}\right)_{\alpha_1}\right]^{-1}d\beta, \tag{50}$$

where
$$f(\beta_1) = i[\beta_0\sin\theta+\{A_0+iB_0/R+A_0'''\delta_1^3/6R^{\frac{3}{2}}+\ldots\}\cos\theta],$$
$$f'''(\beta_1) = i\cos\theta\{A_0'''+A_0^{\text{iv}}\delta_1/R^{\frac{1}{2}}+\ldots\}. \tag{51}$$

The integral in (50) can now be evaluated by steepest descents. This will not be written down explicitly as the calculation is similar to case 1. The asymptotic evaluation of (50), retaining only the viscous effects in the exponential term, gives

$$\eta \sim \frac{G_0}{\pi\rho g l^2}\frac{3^{\frac{1}{6}}}{2\sqrt{2}}\Gamma(\tfrac{1}{3})\,r^{-\frac{1}{3}}\sin(\tfrac{1}{2}r\sqrt{3})\exp(-9r/R\sqrt{2}) \tag{52}$$

as the wave height along $\theta = \pm\theta_c$. The error terms in (52) are of order $r^{-\frac{2}{3}}$ and $r^{\frac{1}{3}}R^{-\frac{1}{2}}$. The result (52) corresponds to an expansion of (39) for small $(\theta-\theta_c)$. The viscous decay factor is obtained from the exp $(-ir\nu)$ term since on $\theta = \pm\theta_c$, $u_0^+ = u_0^-$, and

$$\exp[ir\{g_1(u_0^+)+g_1(u_0^-)\}/2R] = \exp\{-rR^{-1}B(\beta_c)\cos\theta_c\}. \tag{53}$$

4. The viscous decay rate

The exponential factor $\exp(-rR^{-1}B_0\cos\theta)$ has been found as the major effect of the viscosity on the trailing-wave system. The viscous decay rate for $\theta = 0$ agrees with the two-dimensional result. An attenuation length L, defined by

$$L = Rl(B_0\cos\theta)^{-1}, \tag{54}$$

gives a decay factor e^{-1} for each increase L in distance downstream.

The variation of L with θ on the transverse and diverging waves is found by obtaining β_0 as a function of θ from (33). This is most easily done through (27), giving

$$A_0^2 = \tfrac{1}{8}\{4+\cot^2\theta\pm\cot^2\theta(1-8\tan^2\theta)^{\frac{1}{2}}\} \tag{55}$$

and then using (28) for $B_0(\theta)$. The positive sign in (55) applies to the diverging waves, and the negative sign applies to the transverse waves. The factor

$[B_0 \cos\theta]^{-1}$ in L is found to vary only slightly from the value $\frac{1}{4}$ over most of the transverse wave. However, near the critical angle and on the diverging waves, there is considerably more damping. This is shown in table 1 where a number of values of $[B_0 \cos\theta]^{-1}$ are given for θ close to θ_c.

	$[B_0 \cos\theta]^{-1}$	
θ	Transverse	Diverging
12°	0·241	0·015
14°	0·236	0·029
16°	0·227	0·051
18°	0·209	0·087
19°	0·185	0·119
19·5°	0·157	0·157

TABLE 1

5. Concluding remarks

The expansion scheme outlined in §3 yields expressions for the wave height valid for large Reynolds number and for large distance downstream. The expansion is not uniformly valid for large r. That is, the orders of magnitude of r and R are to be such that the errors quoted remain small. It is seen that the exponential decay rate is the main effect of viscosity and the damping of the transverse wave system varies little with angle from a two-dimensional value, whereas the diverging system is more heavily damped. In fact, the diverging waves in the region near $\theta = 0$ are severely damped by viscous forces and the result of infinite wave height associated with a point distribution of pressure in the inviscid case is not achieved.

It must be pointed out that the analysis deals with only part of the problem of the effect of viscosity on ship waves. Problems requiring further study include the effect of the boundary layer on wave making and the interaction of waves with a wake region. The results obtained here establish the relative effects of radial and viscous attenuation for ship waves and may be of use in predicting the wave height for an inviscid model from measurements made in an experiment. Such measurements are used in making wave-drag calculations.

REFERENCES

CRAPPER, G. D. 1964 Surface waves generated by a travelling pressure point. *Proc. Roy. Soc.* A, **282**, 547.

LIGHTHILL, M. J. 1960 Studies on magnetohydrodynamic waves and other anisotropic wave motions. *Phil. Trans.* A, **252**, 397.

PETERS, A. S. 1949 A new treatment of the ship wave problem. *Comm. Pure Appl. Math.* **2**, 123.

THOMSON, W. (Lord Kelvin) 1891 *Popular Lectures and Addresses*, vol. **3**, p. 481. London: MacMillan.

URSELL, F. 1960 On Kelvin's ship wave pattern. *J. Fluid Mech.* **8**, 418.

WEHAUSEN, J. V. & LAITONE, E. 1960 Surface waves. *Handbuch der Physik*, vol. **9**. Berlin: Springer.

WU, T. Y. & MESSICK, R. E. 1958 Viscous effect on surface waves generated by steady disturbances. *Caltech. Rep.* no. 85–8.